Pre-class Warm-up

Do you remember what the gradient of a function is?

Can we do grad f if

a. f : R -> R^2 ? e.g. f(t) = (t,t^2)

b. f : R^2 -> R ? e.g. f(s,t) = s + t^2

c. f : R^3 -> R ? e.g. f(x,y,z) = xy + z

Solution to b. $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial s} \\ \frac{\partial f}{\partial t} \end{bmatrix} = \begin{bmatrix} 1 \\ 2t \end{bmatrix} = \begin{pmatrix} 1 & 2t \end{pmatrix}$

$Df = \begin{bmatrix} \frac{\partial f}{\partial s} & \frac{\partial f}{\partial t} \end{bmatrix}$

Another way to ask this: Find grad f.

Second question: is there a difference between grad f and Df (the derivative of f)?

a. They are pretty much the same

b. They are different but some similarities

c. Completely different.

Section 2.6: Directional derivatives and the gradient

We learn:

- What is the directional derivative of a function f : R^3 -> R?
  (It could be f: R^n -> R)

- The connection between the gradient and the directional derivative.

- The gradient points in the direction of greatest increase of f .

- The gradient points perpendicular to level sets.

- Using this to compute tangent planes etc.

The directional derivative

Suppose we have a function f : R^n -> R.
Let v be a vector of length 1 and a any vector in R^n .
The directional derivative of f at a in the direction v is

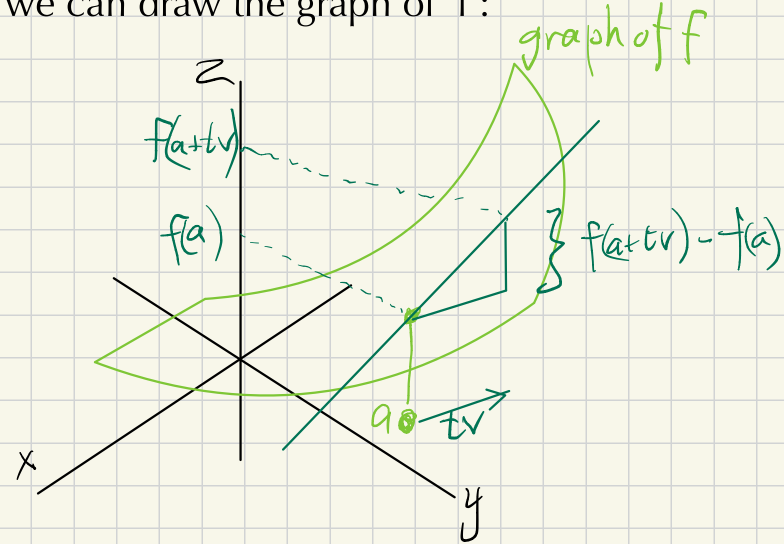
$$\lim_{t \to 0} \frac{f(a+tv) - f(a)}{t}$$

Note $\|tv\| = t$ if $t > 0$.

What if v wasn't a unit vector?

We get confused.

The book sticks to n = 3. When n = 2 we can draw the graph of f :



The directional derivative is the slope of the graph in direction v .

Theorem 12 Let f : R^n -> R, a and v vectors in R^n with v of length 1.
The directional derivative equals

matrix multip.

Df(a) v = grad f (a) • v

good notation for dir. der.

dot product

If these are written out fully it looks like:

$$= \frac{\partial f}{\partial x}\bigg|_a v_1 + \frac{\partial f}{\partial y}\bigg|_a v_2 + \frac{\partial f}{\partial z}\bigg|_a v_3$$

Proof. We can use the chain rule.

Let $c(t) = a + tv$ so the dir. der is $\frac{d\, f\circ c}{dt}\bigg|_{t=0} = Df(c(0)) \cdot Dc(0)$

$$= \begin{bmatrix} \frac{\partial f}{\partial x}\Big|_a & \frac{\partial f}{\partial y}\Big|_a & \frac{\partial f}{\partial z}\Big|_a \end{bmatrix} c'(t) = Df(a)\, v.$$

$c'(t) = v$

Example: Compute the directional derivative of
f(x,y) = x^2 + xy
in the direction of (3/5,4/5). at $a = (1,2)$

Solution $Df = [2x+y \quad x] = [4, 1]$ at a

Answ. $[4, 1] \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \frac{12}{5} + \frac{4}{5} = 3\frac{1}{5}$

Quick question:

Is $\partial f / \partial x$ any of the following?

a. a unit vector

b. a directional derivative, in direction x

c. a directional derivative, in direction y

Theorem 13 If grad f (a) $\neq 0$ then grad f (a) points in the direction along which f is increasing the fastest.

Proof The slope in direction v is

$$\nabla f(a) \cdot v = \|\nabla f(a)\| \|v\| \cos\theta$$

θ = angle between v and $\nabla f(a)$.

This biggest when $\theta = 0$.

The slope is biggest when v points in the direction of $\nabla f(a)$.

Theorem 14 If S is a level set of f defined by f(a) = k then grad f (a) is perpendicular to S.

Proof. If v points in the direction of a level set, the slope of f in direction v is 0.

This means

$\nabla f(a) \cdot v = 0$, so v and $\nabla f(a)$ are perpendicular.

This means we can compute tangent planes to surfaces, because grad f is a normal vector

Example. Compute the tangent plane to the surface x^2 + y^2 + z = 7 at the point (2,1,2).

Solution. The surface is a level set of the function $f(x,y,z) = x^2 + y^2 + z$

$\nabla f = (2x \quad 2y \quad 1) = (4 \quad 2 \quad 1)$ at $(2 \; 1 \; 2)$

The plane has equation

$4x + 2y + z = D$.

Substitute $(x,y,z) = (2,1,2)$

to get $8 + 2 + 2 = D = 12$

Answ. $4x + 2y + z = 12$.

Like qn 4. You are walking on the graph of f(x,y) = xy^2 + y + 3 standing at the point (2,1,6). Find an (x,y)-direction you should walk in to stay at the same level.

Solution. $\nabla f = (y^2, 2xy+1)$

When $(x,y) = (2,1)$

$\nabla f(2,1) = (1, 5)$.

This is perp. to the level direction.

Solve $(v_1, v_2) \cdot (1, 5) = 0$

Answ $(5, -1)$